Characteristic-functional approach to the study of stochastic fluctuations in a model of ionization growth in a discharge gap

Javier E. Vitela
Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México 04510, México, Distrito Federal, México

Lorena Zogaib
Instituto de Investigaciones en Materiales, Universidad Nacional Autónoma de México 04510, México, Distrito Federal, México
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A characteristic-functional approach is introduced to study the space and time stochastic fluctuations of the electron population in a simple one-dimensional model of ionization growth. Two electron sources are considered: (a) ionization by direct collisions and (b) photoemission at the cathode due to de-excitation of atoms. The motion of the ions is neglected and the electrons are assumed to move with constant drift velocity. An equation for the characteristic functional $G[\theta(x),t]=\langle \exp[i\int_0^L dx\,\theta(x)n(x,t)]\rangle$ is obtained, where $n(x,t)$ is the electron density, and $\theta(x)$ is a conjugate function; from this, equations for the moments, e.g., the average density and the density-density correlation function, can easily be derived. Similarly to using the method of compounding moments, this technique avoids the use of a probability in function space; however, it has the benefit that the motion in configuration space may be incorporated self-consistently, whether the motion is free streaming or diffusion. Numerical examples are used to illustrate the time behavior of the mean total population in the gap, which shows a good agreement with previous results; in addition, we analyze the time evolution of the associated electron mean density, the density-density correlation function, and the fluctuations around the mean population.

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I. INTRODUCTION

The statistics of the evolution of the electron population in an electric discharge between parallel electrodes has been a subject of research that spans several decades [1–7]. The works have dealt only with the total-electron population; some are devoted basically to the task of finding the probability that the discharge should have become extinguished at a particular time after the release of an initial electron at a certain position between the electrodes [1,2,7]; others have been concerned with the frequency spectrum of the fluctuations in the electric current, both theoretically and experimentally, under a variety of conditions [6]. It has long been known that when the voltage between the electrodes exceeds a certain value, it is possible to initiate a self-sustained gas discharge; the threshold voltage at which this phenomenon can occur is called the breakdown potential, and the transition from a nonconducting to a conducting state is called a breakdown [1]. Since most of the processes involved in a gas discharge (e.g., ionization by collision and the emission of secondary electrons at the cathode) are random phenomena, an initial electron liberated in the gas by some means does not necessarily lead to a breakdown, even though the voltage applied between the electrodes exceeds the breakdown potential. Hence, it is natural to study the evolution of the discharge from the point of view of the theory of stochastic processes.

In many physical situations, the approach to the study of the stochastic behavior of populations is determined by the homogeneity conditions prevailing in the system under consideration. In a homogeneous system, for instance, any fluctuation in the total population spreads throughout the entire system before it disappears; thus, in principle, the total population at a given time completely determines its future evolution. On the contrary, fluctuations in nonhomogeneous systems are local phenomena which can only be described by introducing the spatial density as a random object [8,9].

An electrical discharge does not constitute a homogeneous system, since the mere information of an initial number of electrons cannot determine uniquely the probability of having a given population at a later time—that is, the total number of electrons generated by a single initial electron is highly sensitive to its initial location inside the gap. Thus, previous works [1-7] have been based on heuristics derivations of evolution equations for the probability distribution of the total-electron population. In this paper, a Markov process for the evolution of an electric discharge is constructed by accounting for the space distribution of electrons at each time. Our aim here is to study the space-time stochastic behavior of the electron population between parallel plates in a simple model of electric discharge, based on the general theory of Markov processes.

A widely used technique to study fluctuations in

nonhomogeneous systems is the method of compounding moments, which avoids the use of a probability in function space [8]. It consists simply in subdividing the configuration space into cells; calling n_λ the number of particles in cell λ, a master equation for $P(\{n_\lambda\},t|\{n_\lambda^{(0)}\})$, the conditional probability for the distribution $\{n_\lambda\}$ at time t given a distribution $\{n_\lambda^{(0)}\}$ at $t=0$ is constructed. From this equation, the evolution equations for the first- and second-order moments of P can be obtained; the resulting expressions are divided by the width of the cells to express them as continuous functions of space. In general, this is a three-step method: first, a master equation is derived for separate unconnected cells to obtain time-evolution equations for the desired moments, where the motion in the configuration space is neglected; this is done by considering exclusively events that create or annihilate a particle. The corresponding equations for the continuum are obtained, then, by taking the limit when the width of the cells goes to zero. The second step also consists in constructing a master equation for the discretized system, but taking into account only the motion of the particles from one cell to another one, which is described through transition probabilities per unit time; the desired moments are obtained from this equation and expressed in continuous notation by means of a continuous transition probability per unit time. Assuming that the size of the jumps is small, an expansion of this transition probability per unit time is performed to obtain the appropriate evolution equations for the moments. Finally, in the third stage the effect of the previous steps is combined to obtain the desired equations for the moments.

Here we approach the problem in a way similar to that of the compounding moments method, but the master equation is constructed including simultaneously all processes responsible for changing the population in each cell; the motion in the configuration space is accounted for by prespecified transition probabilities per unit time for jumps into neighboring cells, whether the motion is diffusive, free streaming, or some other type. Instead of deriving equations for the first few moments of $P(\{n_\lambda\},t|\{n_\lambda^{(0)}\})$, we find the evolution equation for its associated characteristic function and by taking the limit when the number of cells goes to infinity and their width goes to zero, an equation for the characteristic functional is obtained. Finally, functional derivatives of the latter yield the evolution equations for the moments in the continuous model.

The characteristic-functional technique has the advantage that the motion in the configuration space, as well as the boundary conditions when the system is finite, appear naturally in the equations for the moments. Furthermore, in this approach the limit to the continuous model is only performed once, when deriving the equation for the characteristic functional, while in the method of compounding moments, the continuous limit has to be taken in each one of the derived equations for the moments.

For our purpose, we adopt a simple one-dimensional model of an electric discharge between parallel plates, where the gap extends from $x=0$ to $x=L$. There exists a uniform electric field $\mathcal{E}$ between the electrodes and a pressure $\mathcal{P}$ of the filling gas. It is assumed that the electrons generated by the ionization processes reach equilibrium at the same point where they are produced. In addition to the electrons, a complete description should account for the production of ions, metastable atoms, and radicals, as well as any possible chemical reaction between them [10–12]. To simplify the problem, we restrict ourselves to the electron population only and consider the following two ionization processes: (i) an electron may suffer an ionizing collision, producing an additional electron at the same position, with probability per unit time a, and (ii) an electron can also excite an atom to emit a photon after a negligible delay, which may produce an electron emitted from the cathode, and the whole process is taken into account by a probability per unit time b. All electrons are assumed to move at a constant drift velocity μ, regardless of the position or any collision they may suffer; the velocity distribution of the background particles is assumed to remain unaltered, and any motion of ions, radicals, and metastable species is neglected. Space-charge effects, electron-ion recombination, and electron attachment and detachment are not included. With these assumptions, it is reasonable to consider the electron-density distribution to constitute a Markov process.

Since we are concerned here with a limited aspect of the processes involved in an electrical discharge, the theoretical model adopted in this work makes no attempt to include energy- and momentum-conservation equations. However, we assume that the velocities of the electrons always follow an equilibrium distribution at a constant temperature kT. This assumption is expected to be valid whenever the drift velocity μ is much smaller than the thermal speed of the electrons, so that its effect on the velocity distribution is negligible. A constant drift velocity is achieved when the mean frictional force due to collisions with the background particles compensates the accelerating force of the external electric field, so that $\mu\sim\mathcal{E}$. This, together with the equilibrium assumption, leads to the condition $e\mathcal{E}/\sigma\mathcal{P}\ll 1$, where σ is the average cross section for momentum transfer and e is the electronic charge. Implicit in the former arguments are the conditions that space and time scales are much larger than the mean free path $kT/\sigma\mathcal{P}$ and the mean collision time $(mkT)^{1/2}/\sigma\mathcal{P}$, respectively.

The rest of the paper is organized as follows: In Sec. II the master equation for the discretized model and the equation for the associated characteristic function are derived. Section III is concerned with the derivation of the characteristic functional, as the limit to the continuum of the characteristic function derived in Sec. II, and equations for the mean number density of electrons and the density-density correlation function are found. In Sec. IV the solution to the partial integro-differential equation for the mean density is obtained, as well as the expression for the mean total number of electrons within the electrodes. In Sec. V, the solution to the equation for the density-density correlation function is analyzed in general and discussed in detail for a particular discharge that tends to extinguish in time. Finally, Sec. VI contains concluding remarks.

II. DISCRETE MODEL: THE MASTER EQUATION AND THE EQUATION FOR THE CHARACTERISTIC FUNCTION

We first discretize the configuration space in a number K of cells of width $\Delta x = L/K$ and focus attention on the distribution $\{n_1, n_2, \ldots, n_K\}$ of electrons into the cells as a function of time (see Fig. 1). The random state vector $\mathbf{n} \equiv \{n_1, n_2, \ldots, n_K\}$ constitutes a Markov process, whose statistical properties are completely described by the conditional probability $P(\mathbf{n}, t|\mathbf{n}_0)$. The Markovian property follows from the fact that the evolution in the configuration space of a given distribution of electrons depends only on the distribution itself and not on how it was reached, since the medium is assumed to provide no memory. The time evolution for the conditional probability P satisfies, then, a master equation of the form

$$\frac{d}{dt}P(\mathbf{n},t|\mathbf{n}_0) = \sum_{\mathbf{m}} \{Q(\mathbf{m}|\mathbf{n})P(\mathbf{m},t|\mathbf{n}_0) - Q(\mathbf{n}|\mathbf{m})P(\mathbf{n},t|\mathbf{n}_0)\} , \quad (1)$$

where $Q(\mathbf{m}|\mathbf{n})$ is the transition probability per unit time from a distribution $\mathbf{m}$ to a distribution $\mathbf{n}$. The solution to this equation satisfies the initial condition

$$P(\mathbf{n},0|\mathbf{n}_0) = \delta_{n_1,n_{10}}\delta_{n_2,n_{20}}\cdots\delta_{n_K,n_{K0}} , \quad (2)$$

as well as the normalization condition

$$\sum_{\mathbf{n}} P(\mathbf{n},t|\mathbf{n}_0) = 1, \quad \forall t .$$

According to the model adopted here, multiplication of electrons in the discharge is taken into account by the probabilities a and b, as discussed in the Introduction. The constant drift velocity μ of the electrons is accounted for by assuming that the free streaming is also a random process, with probability per unit time $\mu/\Delta x$. This expression is justified by the assumption that in a given cell there is no preferred location for an electron, so that the probability density to have an electron in a certain position within the cell is simply $1/\Delta x$; hence, the probability for an electron in a given cell to jump into the next cell within the time interval dt becomes $\mu\, dt/\Delta x$, from which the desired expression for the transition probability per unit time follows [9,13].

In order to obtain the transition probability per unit time Q, the following processes are considered:

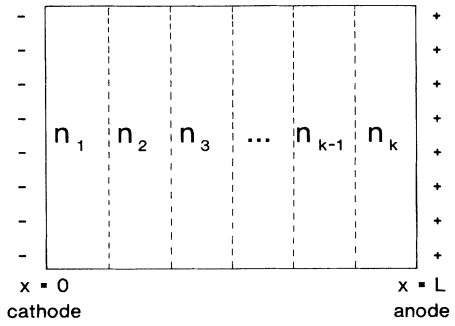

FIG. 1. Diagram of the simplified one-dimensional representation of an electric discharge between parallel plates, showing the cell partitions used for the discretized model.

(i) Primary production. An electron in cell λ ($\lambda = 1, \ldots, K$) may suffer an ionizing collision, producing an additional electron, with probability per unit time a, generating a transition

$$\{m_1, \ldots, m_\lambda, \ldots, m_K\} \rightarrow \{m_1, \ldots, m_\lambda + 1, \ldots, m_K\} .$$

(ii) Secondary production. An electron in any cell may suffer a collision, exciting an atom to emit a photon which may produce a photoelectron from the cathode, with probability per unit time b, producing a transition

$$\{m_1, \ldots, m_\lambda, \ldots, m_K\} \rightarrow \{m_1 + 1, \ldots, m_\lambda, \ldots, m_K\} .$$

(iii) Free-streaming motion. An electron in cell λ ($\lambda = 1, \ldots, K-1$) may jump into the next cell, with probability per unit time $\mu/\Delta x$, generating a transition

$$\{m_1, \ldots, m_\lambda, m_{\lambda+1}, \ldots, m_K\} \rightarrow \{m_1, \ldots, m_\lambda - 1, m_{\lambda+1} + 1, \ldots, m_K\} .$$

(iv) Losses in the anode. An electron in cell K arrives to the anode, where it is absorbed with a probability per unit time $\mu/\Delta x$, producing a transition

$$\{m_1, \ldots, m_\lambda, \ldots, m_K\} \rightarrow \{m_1, \ldots, m_\lambda, \ldots, m_K - 1\} .$$

Once processes (i)–(iv) are all taken into account, the transition probability per unit time is found to be

$$Q(\mathbf{m}|\mathbf{n}) = a \sum_{j=1}^{K} m_j (\delta_{n_1,m_1}\cdots\delta_{n_K,m_K})\Big|_{m_j \rightarrow m_j+1} + b\left[\sum_{j=1}^{K} m_j\right](\delta_{n_1,m_1+1}\cdots\delta_{n_K,m_K}) + \left[\frac{\mu}{\Delta x}\right]\sum_{j=1}^{K-1} m_j (\delta_{n_1,m_1}\cdots\delta_{n_K,m_K})\Big|_{m_j \rightarrow m_j-1,\, m_{j+1} \rightarrow m_{j+1}+1} + \left[\frac{\mu}{\Delta x}\right] m_K(\delta_{n_1,m_1}\cdots\delta_{n_K,m_K-1}) , \quad (3)$$

where the notation $m_j \to m_j - 1$ indicates replacement of m_j by $m_j - 1$ in the corresponding Kronecker's δ, and a similar notation is used for m_{j+1}. Substituting this expression in Eq. (1) yields the master equation for this model.

Among several equivalent representations of the conditional probability function [8,14,15], $P(\mathbf{n},t|\mathbf{n}_0)$, the most convenient for our purpose is that of the characteristic function

$$C(\theta;t)=\sum_{n_1,\ldots,n_K=0}^{\infty} e^{i\sum_{j=1}^{K}\theta_j n_j} P(\mathbf{n},t|\mathbf{n}_0) , \tag{4}$$

where $\theta=\{\theta_1,\ldots,\theta_K\}$ is the conjugate state vector, with θ_i a real auxiliary variable with range $(-\infty,\infty)$. This function contains the same information as the probability distribution, but it is easier to handle in general. The evolution equation for the characteristic function, obtained from Eqs. (1), (3), and (4), is

$$-i\frac{\partial}{\partial t}C(\theta;t)=\sum_{j=1}^{K}\left\{\left[a+b+\frac{\mu}{\Delta x}\right]-ae^{i\theta_j}-be^{i\theta_1} -\frac{\mu}{\Delta x}e^{i(\theta_{j+1}-\theta_j)}\right\}\frac{\partial}{\partial\theta_j}C(\theta;t) , \tag{5}$$

with $\theta_{K+1}\equiv 0$. The equations for the moments of the conditional probability function, e.g., $\langle n_j(t)\rangle$, $\langle n_j(t)n_l(t)\rangle$, $\forall j,l=1,\ldots,K$, can easily be derived from Eq. (5) by simple derivative operations with respect to the variables θ_j; thus, for the first two moments we get

$$\frac{d}{dt}\langle n_j\rangle = a\langle n_j\rangle + b\delta_{j,1}\sum_i\langle n_i\rangle + \left[\frac{\mu}{\Delta x}\right]\sum_i(\delta_{i+1,j}-\delta_{i,j})\langle n_i\rangle \tag{6}$$

and

$$\begin{aligned}\frac{d}{dt}\langle n_j n_l\rangle =& a\{2\langle n_j n_l\rangle+\delta_{j,l}\langle n_j\rangle\}+b\left\{\delta_{j,1}\sum_{i=1}^{K}\langle n_i n_l\rangle+\delta_{l,1}\sum_{i=1}^{K}\langle n_i n_j\rangle+\delta_{j,1}\delta_{l,1}\sum_{i=1}^{K}\langle n_i\rangle\right\}\\ &-\frac{\mu}{\Delta x}[\langle n_l n_j\rangle-\langle n_l n_{j-1}\rangle]-\frac{\mu}{\Delta x}[\langle n_l n_j\rangle-\langle n_j n_{l-1}\rangle]\\ &-\frac{\mu}{\Delta x}\delta_{j,1}\langle n_l n_0\rangle-\frac{\mu}{\Delta x}\delta_{l,1}\langle n_j n_0\rangle+\frac{\mu}{\Delta x}\delta_{K,j}\delta_{K,l}\langle n_K\rangle\\ &+\frac{\mu}{\Delta x}\sum_{m=1}^{K-1}[\delta_{j,m+1}\delta_{l,m+1}-\delta_{j,m+1}\delta_{l,m}-\delta_{j,m}\delta_{l,m+1}+\delta_{j,m}\delta_{l,m}]\langle n_m\rangle \quad \forall j,l=1,\ldots,K .\end{aligned} \tag{7}$$

Equations (6) and (7) constitute a set of $K(K+1)$ coupled differential equations describing the time evolution of the mean and the correlation function of the electron population in the cells.

III. THE CONTINUOUS DESCRIPTION: CHARACTERISTIC-FUNCTIONAL APPROACH

The discrete description of Sec. II provides an adequate representation of the process, when the size of each cell is so small that inside each of them the homogeneity condition mentioned in Sec. I prevails. However, this artificial discretization of space introduces an inconvenient high-dimensional conjugate state vector to describe the time evolution of the characteristic function in Eq. (5). Instead, we cast this equation in a form in which the set of conjugate variables $\{\theta_j\}$ occurs as a continuously varying function of space, by taking the limit when the number of cells goes to infinity and their width Δx goes to zero; in this limit, the characteristic function (4) becomes the characteristic functional [16]

$$\begin{aligned}C[\theta(x),t]\equiv&\langle e^{i\int_0^L dx'\theta(x')n(x')}\rangle\\ =&\lim_{\substack{K\to\infty\\ \Delta x\to 0}}\left\langle\exp\left\{i\sum_{j=1}^{K}\theta(x_j)n(x_j)\Delta x\right\}\right\rangle ,\end{aligned} \tag{8}$$

where θ_j has been replaced by $\theta(x_j)$ and n_j by $n(x_j)\Delta x$. The time dependence of the characteristic functional is due to the time dependence of the probability distribution, as shown in Eq. (4). Thus, taking this limit in Eq. (5) and assuming that $\theta(x)$ is a continuous function with a first derivative, we get the following equation for the characteristic functional:

$$-i\frac{\partial}{\partial t}C[\theta,t]=\int_0^L dx\left\{\left[a(1-e^{i\theta(x)})+b(1-e^{i\theta(0)})-i\mu\frac{d\theta}{dx}\right]\frac{\delta C[\theta,t]}{\delta\theta(x)dx}\right\} , \tag{9}$$

where

$$\frac{\delta C[\theta(x),t]}{\delta\theta(x')dx'}\equiv\lim_{\epsilon\to 0}\frac{d}{d\epsilon}C[\theta(x)+\epsilon\delta(x-x'),t] \tag{10}$$

is the functional derivative of C [17]. We shall point out here that the term

$$(1-e^{-i\theta(x)})\frac{\delta C}{\delta\theta(x)dx}\bigg|_{x=L}$$

has been neglected in Eq. (9) for consistency with the assumption that $\theta(x)$ is continuous and $\theta(L^+)\equiv 0$.

Equation (9) can be cast in a form more convenient to our purpose by integrating by parts the term containing $d\theta/dx$, giving

$$-i\frac{\partial}{\partial t}C[\theta,t]=\int_0^L dx\left\{[a(1-e^{i\theta(x)})+b(1-e^{i\theta(0)})]\frac{\delta C[\theta,t]}{\delta\theta(x)dx}+i\mu\theta(x)\frac{\partial}{\partial x}\left[\frac{\delta C[\theta,t]}{\delta\theta(x)dx}\right]\right\}+i\mu\left[\theta(x)\frac{\delta C[\theta,t]}{\delta\theta(x)dx}\right]_{x=0}. \tag{11}$$

Time-evolution equations for the mean electron density and the density-density correlation function can be obtained now by using the relations

$$\left.\frac{\delta C[\theta,t]}{\delta\theta(x)dx}\right|_{\theta=0}=i\langle n(x,t)\rangle \tag{12}$$

and

$$\left.\frac{\delta^2 C[\theta,t]}{\delta\theta(x')dx'\delta\theta(x'')dx''}\right|_{\theta=0}=-\langle n(x',t)n(x'',t)\rangle , \tag{13}$$

respectively.

The average $\langle n(x,t)\rangle$ provides a local description of the mean density at each position x in the gap, at a time t, whereas $\langle n(x',t)n(x'',t)\rangle$ accounts for statistical correlations between the microscopic electron densities at two different positions x' and x'', at the same time; the study of the latter is important due to the nonhomogeneous character of the density fluctuations in the discharge gap. The mean total population in the system is calculated by integrating the mean density along the gap length, and the fluctuations around this quantity can be estimated by calculating its variance from the density-density correlation function.

From Eqs. (11) and (12) the mean electron density is found to satisfy the partial integro-differential equation

$$\frac{\partial\rho(x,t)}{\partial t}+\mu\frac{\partial}{\partial x}\rho(x,t)=a\rho(x,t)+b\delta(x)\int_0^L dy\,\rho(y,t)-\mu\delta(x)\rho(x,t) , \tag{14}$$

where the definition

$$\rho(x,t)\equiv\langle n(x,t)\rangle \tag{15}$$

has been introduced. The left-hand side (LHS) of Eq. (14) is the material derivative of the mean number density; on the right-hand side (RHS), the first two terms are the sources due to primary and secondary production, respectively, and the last term represents a negative boundary source whose role is to discard mathematically any contribution to the electron population coming from the left-hand side of the cathode, since this is physically not possible.

According to Eq. (13), the second-order functional derivative of Eq. (11) yields the equation for the density-density correlation function. We shall use, instead, the related quantity

$$R(x',x'',t)\equiv\langle n(x',t)n(x'',t)\rangle-\langle n(x',t)\rangle\langle n(x'',t)\rangle , \tag{16}$$

i.e., the autocorrelation function of the density fluctuations; this function is found to satisfy the partial integro-differential equation

$$\frac{\partial R(x',x'',t)}{\partial t}+\mu\frac{\partial}{\partial x'}R(x',x'',t)+\mu\frac{\partial}{\partial x''}R(x',x'',t)=2aR(x',x'',t)+a\delta(x'-x'')\rho(x',t)+b\left[\delta(x')\int_0^L dy\,R(y,x'',t)+\delta(x'')\int_0^L dy\,R(x',y,t)\right]+b\delta(x')\delta(x'')\int_0^L dy\,\rho(y,t)-\mu[\delta(x')+\delta(x'')]R(x',x'',t) , \tag{17}$$

where Eq. (14) has also been used. We notice here that the last two terms in Eq. (7) do not have their equivalent expressions in Eq. (17). These terms appear in Eq. (7) due to the assumption that the motion from one cell to a neighboring one is a stochastic process with a given transition probability per unit time. By assuming in the continuous model that $\theta(x)$ is continuous, we have removed the stochasticity of the motion in the configuration space; in fact, if $\exp\{i(\theta_{j+1}-\theta_j)\}$ in Eq. (5) is replaced by $1+i(\theta_{j+1}-\theta_j)$, this randomness is also removed from the discrete model.

The solutions to Eq. (14) for the number density and to Eq. (17) for the density-density correlation function are analyzed in Secs. IV and V. As a final remark, it is worth mentioning that the evolution equations for the moments of higher order can also be obtained easily by taking appropriate functional derivatives of Eq. (11).

IV. MEAN BEHAVIOR OF THE ELECTRON DENSITY AND THE TOTAL NUMBER OF ELECTRONS

In order to obtain the mean electron density $\rho(x,t)$, it is convenient to perform a Laplace transform in the time domain of Eq. (14), that is,

$$\frac{\partial}{\partial x}\tilde{\rho}(x,s)+\frac{s-a}{\mu}\tilde{\rho}(x,s)$$
$$=\frac{1}{\mu}\rho(x,0)+\frac{b}{\mu}\delta(x)\int_0^L dy\,\tilde{\rho}(y,s)-\delta(x)\tilde{\rho}(0,s)\ , \quad (18)$$

where $\tilde{\rho}(x,s)$ denotes the Laplace transform of $\rho(x,t)$ [18],

$$\tilde{\rho}(x,s)\equiv\int_0^\infty dt\,\rho(x,t)e^{-st}\ . \quad (19)$$

The solution to Eq. (18) can be shown to be

$$\tilde{\rho}(x,s)=\tilde{\rho}(0,s)[1-\theta(x)]e^{-(s-a)x/\mu}$$
$$+\frac{1}{\mu}\int_0^x dx' e^{-[(s-a)/\mu](x-x')}\rho(x',0)$$
$$+\frac{b}{\mu}\tilde{N}(s)e^{-[(s-a)/\mu]x}\theta(x)\ , \quad (20)$$

where we have defined

$$\tilde{N}(s)\equiv\int_0^L dy\,\tilde{\rho}(y,s)\ , \quad (21)$$

which is the Laplace transform of the total number of electrons in the gap; in Eq. (20), and thereafter, we adopt the following definition of the Heaviside step function:

$$\theta(x)=\begin{cases}0\ , & x\le 0\\ 1\ , & x>0\ .\end{cases} \quad (22)$$

Since the first term on the RHS of Eq. (20) arises from a nonphysical situation, as pointed out in the discussion of Eq. (14), it will be omitted from now on; thus, the electron mean density throughout the interval of interest is given by

$$\tilde{\rho}(x,s)=\frac{b}{\mu}\tilde{N}(s)e^{-[(s-a)/\mu]x}\theta(x)$$
$$+\frac{1}{\mu}\int_0^x dx' e^{-[(s-a)/\mu](x-x')}\rho(x',0)\ . \quad (23)$$

From this, the boundary condition

$$\tilde{\rho}(0^+,s)=\frac{b}{\mu}\tilde{N}(s) \quad (24)$$

follows.

Equations (23) and (24) do not yet constitute a closed system, since these still depend on the total number of electrons, which has not been determined so far. To proceed further, we integrate Eq. (18) from $x=0$ to L and solve for $\tilde{N}(s)$, to obtain

$$\tilde{N}(s)=[N(0)-\mu\tilde{\rho}(L,s)]/(s-a-b)\ . \quad (25)$$

We now evaluate Eq. (23) at $x=L$ and solve again for $\tilde{N}(s)$ to get

$$\tilde{N}(s)=\frac{\mu}{b}[\tilde{\rho}(L,s)-\tilde{G}(L,s)]e^{(s-a)L/\mu}\ , \quad (26)$$

with

$$\tilde{G}(x,s)\equiv\frac{1}{\mu}\int_0^x dx' e^{-(s-a)(x-x')/\mu}\rho(x',0)\ . \quad (27)$$

Eliminating, then, $\tilde{\rho}(L,s)$ from Eqs. (25) and (26), we obtain

$$\tilde{N}(s)=\frac{N(0)-\mu\tilde{G}(L,s)}{(s-a-b)\left[1+\frac{b}{s-a-b}e^{-(s-a)L/\mu}\right]}\ . \quad (28)$$

Finally, substitution of this latter expression in Eq. (23) yields a closed equation for $\tilde{\rho}(x,s)$, i.e.,

$$\tilde{\rho}(x,s)=\tilde{G}(x,s)$$
$$+\frac{b}{\mu}\frac{[N(0)-\mu\tilde{G}(L,s)]e^{-(s-a)x/\mu}}{(s-a-b)\left[1+\frac{b}{s-a-b}e^{-(s-a)L/\mu}\right]}\ . \quad (29)$$

In order to obtain $\rho(x,t)$, we shall perform the inverse Laplace transform of Eq. (29); to this end, we make use of the expansion

$$\left[1+\frac{b}{s-a-b}e^{-(s-a)L/\mu}\right]^{-1}$$
$$=\sum_{n=0}^\infty\frac{(-1)^n b^n}{(s-a-b)^n}e^{naL/\mu}e^{-nLs/\mu}\ , \quad (30)$$

whose validity is proved in Appendix A. Substituting Eq. (30) in Eq. (29), finally, yields the solution to Eq. (14), valid in the interval $(0,L)$, namely,

$$\rho(x,t)=\rho(x-\mu t,0)e^{at}+\frac{b}{\mu}N(0)e^{at}\sum_{n=0}^\infty\frac{(-1)^n b^n}{n!}e^{b[t-(x+nL)/\mu]}[t-(x+nL)/\mu]^n\theta(t-(x+nL)/\mu)$$
$$-\frac{b}{\mu}e^{ax/\mu}\int_0^L dy\,e^{a(L-y)/\mu}\rho(y,0)\sum_{n=0}^\infty\frac{(-1)^n b^n}{n!}(t-[x-y+(n+1)L]/\mu)^n e^{anL/\mu}e^{(a+b)(t-[x-y+(n+1)L]/\mu)}$$
$$\times\theta(t-[x-y+(n+1)L]/\mu)\ . \quad (31)$$

The total number of electrons between the electrodes can be obtained by integrating Eq. (31) or, alternatively, by taking the inverse Laplace transform of Eq. (28). The resulting expression is

$$N(t)=N(0)e^{at}\sum_{n=0}^\infty\frac{(-1)^n b^n}{n!}\left[t-\frac{nL}{\mu}\right]^n e^{b(t-nL/\mu)}\theta(t-nL/\mu)$$
$$-\sum_{n=0}^\infty\frac{(-1)^n b^n}{n!}e^{at}\int_0^L dx\,\rho(x,0)(t-[(n+1)L-x]/\mu)^n e^{b(t-[(n+1)L-x]/\mu)}\theta(t-[(n+1)L-x]/\mu)\ . \quad (32)$$

The long-time limit of the electron density and the total number of electrons, if they exist, are obtained by using the following properties of the Laplace transform:

$$\rho(x,\infty)=\lim_{s\to 0} s\tilde{\rho}(x,s)\ ,\quad N(\infty)=\lim_{s\to 0} s\tilde{N}(s)\ . \tag{33}$$

It follows that the asymptotic value of the total number of electrons is obtained from the limit

$$N(\infty)=\lim_{s\to 0}\frac{s\left[N(0)-\int_0^L dx\, e^{-(s-a)(L-x)/\mu}\rho(x,0)\right]}{(s-a-b)\left[1+\dfrac{b}{s-a-b}e^{-(s-a)L/\mu}\right]}\ ; \tag{34}$$

it is easy to verify that this limit exists only when the parameters of the system satisfy the relation

$$\frac{b}{a}(e^{aL/\mu}-1)=1\ . \tag{35}$$

This latter expression will be referred to as the *critical condition,* or Towsend breakdown criterion [7,10–12]. When this condition is satisfied, the asymptotic values for the mean total number of electrons and the mean number density become, respectively,

$$N(\infty)=\frac{1}{\dfrac{bL}{\mu}e^{aL/\mu}-1}\left[\int_0^L dx\, e^{a(L-x)/\mu}\rho(x,0)-N(0)\right] \tag{36}$$

and

$$\rho(x,\infty)=\frac{b}{\mu}N(\infty)e^{ax/\mu}\ . \tag{37}$$

The critical condition, Eq. (35), can be interpreted as follows: Assume that N_0 initial electrons are liberated at the cathode by some means, e.g., illumination of the cathode with ultraviolet radiation; as they travel to the anode, their average population due to primary ionization increases as $N_0\exp(at)$, which is producing secondary electrons at the cathode, at an average rate $bN_0\exp(at)$. Thus, when the leading electrons reach the anode, at time $t=L/\mu$, they have produced a total of $bN_0[\exp(aL/\mu)-1]/a$ electrons at the cathode, due to secondary ionization. In order to satisfy steady-state conditions, this number has to be equal to N_0, from which the breakdown criterion follows. When this number exceeds N_0, that is, when $(b/a)(e^{aL/\mu}-1)>1$, the mean total electron population will increase in time; on the other hand, if $(b/a)(e^{aL/\mu}-1)<1$, the population will eventually die out. We shall refer to these as supercritical and subcritical conditions, respectively.

Equation (35) is in agreement with the breakdown criteria obtained in previous works [7,10–12], where the coefficient b/a is interpreted as the total number of secondary electrons produced per primary electron; the secondary production may include electrons generated by processes such as the incidence of positive ions and metastable atoms on the cathode, collisions of positive ions with the filling gas, photoionization at the cathode, and photoionization of the gas itself.

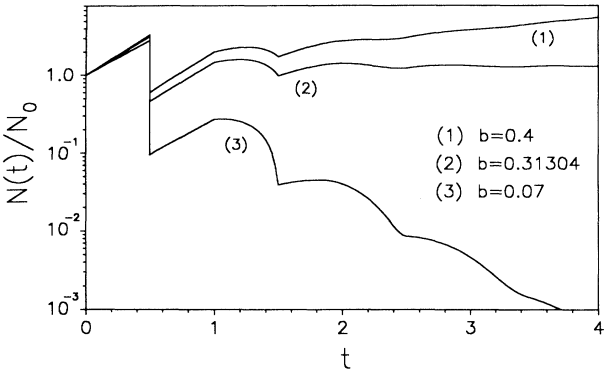

FIG. 2. Time behavior of the mean total-electron population for an initial distribution $\rho(x,0)=N_0\delta(x-x_0)$, where $x_0=0.5$. The plots correspond to different discharges with the same primary production parameter, $a=2.0$. In each case, the secondary electron production parameter b has been chosen to satisfy supercritical (1), critical (2), and subcritical (3) conditions. Dimensionless quantities are used, as defined in the text.

To illustrate the results of this section, let us study the evolution of a deterministic initial-electron distribution of the form $\rho(x,0)=N_0\delta(x-x_0)$, with $x_0=L/2$, representing N_0 electrons located in the middle of the discharge gap. To simplify the notation, the dimensionless quanti-

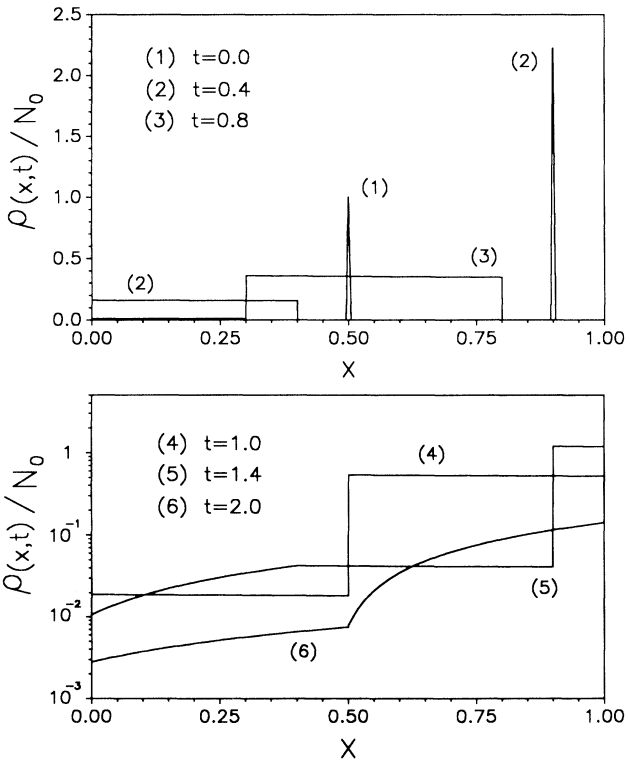

FIG. 3. Normalized mean electron-density distribution corresponding to the subcritical discharge ($a=2.0$, $b=0.07$) of Fig. 2. The plots illustrate the mean electron distribution at several times t. Dimensionless quantities are used, as defined in the text.

ties $\mu t/L \rightarrow t$, $x/L \rightarrow x$, $aL/\mu \rightarrow a$, and $bL/\mu \rightarrow b$ are introduced to denote time, position, and the primary and secondary ionization coefficients, respectively. The space-time evolution of the electron population in the gap is exhibited in Figs. 2 through 5. Three different types of discharges are considered, having the same primary parameter ($a=2.0$), but different secondary ionization coefficients, describing a supercritical discharge ($b=0.4$), a subcritical discharge ($b=0.07$), and the critical discharge ($b=0.313\,04$).

The plots of Fig. 2 show the time behavior of the mean total number $N(t)$, for the three discharges, as obtained from Eq. (32). From these plots we can distinguish three time regions about $t=0.5$ and 1.0, related to the population growth due to secondary production. For $0<t<0.5$, the mean population increases monotonically as $N_0\exp(a+b)t$, in all cases; thus, the contribution from the secondary ionization process to the total production is simply given by the exponential factor $\exp(bt)$, regardless of the type of discharge under consideration. At time $t=0.5$, the initial electrons and their direct primary descendants reach the anode, giving rise to a discontinuity ΔN in the total population, with

$$\Delta N/N=\exp[-b(1-x_0)]$$

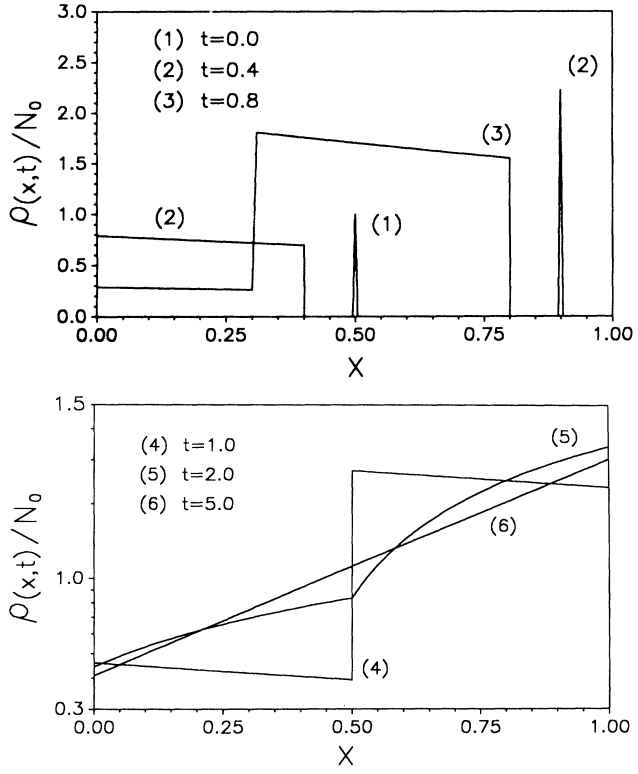

FIG. 4. Normalized mean electron-density distribution corresponding to the critical discharge ($a=2.0$, $b=0.313\,04$) of Fig. 2. The plots illustrate the mean electron distribution at several times t. As time evolves, the electron density tends asymptotically to the stationary exponential distribution, Eq. (37), as shown in the plot labeled (6). Dimensionless quantities are used, as defined in the text.

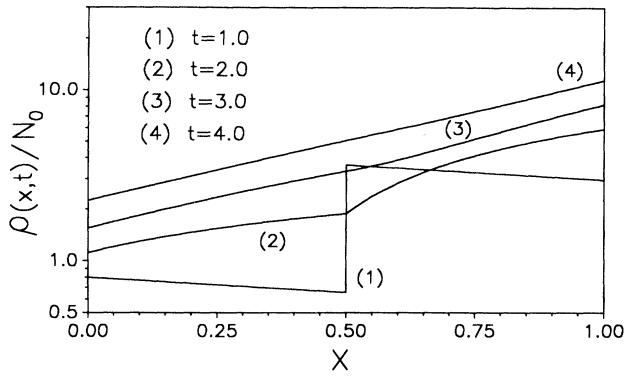

FIG. 5. Normalized mean electron-density distribution corresponding to the supercritical discharge ($a=2.0$, $b=0.4$) of Fig. 2. The plots illustrate the mean electron distribution at several times t. Dimensionless quantities are used, as defined in the text.

for an arbitrary initial position x_0. For times $0.5<t<1.0$, the population grows exponentially again, as

$$N_0(1-e^{-b/2})e^{(a+b)t}\,.$$

The evolution of the discharge at later times, given a fixed value of a, will now depend strongly on the magnitude of the secondary electron production parameter, as shown in the plots of this figure. To further understand the behavior of $N(t)$, it is necessary to consider the internal distribution of electrons in the discharge gap, as follows.

The electron-density distribution at different times is shown in Figs. 3, 4, and 5, for the subcritical, critical, and supercritical discharges, respectively. All cases present a similar behavior for times $0<t<0.5$, namely, the initial δ distribution moves uniformly towards the anode, increasing its average population by a factor e^{at}, due to primary ionization; the δ is followed by a second front of electrons at a distance $\Delta x=0.5$ from it, as can be seen from the plots labeled (2) in Figs. 3 and 4. The second front is produced by the electrons that are released from the cathode due to photoionization. At time $t=0.5$, the leading δ reaches the anode originating the discontinuity in the total population shown in Fig. 2. For times $0.5<t<1.0$, no electron losses are present and the population increases monotonically again; this is due to the fact that during this time interval the second front has not reached the anode yet, as can be seen from the plots labeled (3) in Figs. 3 and 4. This front is composed of a main square-like pulse, traveling uniformly towards the anode, and a tail of electrons that extends to the cathode. At $t=1.0$ the leading edge of the second front reaches the anode, and the electron losses produce a discontinuity in the growth rate of $N(t)$, as shown in Fig. 2. After $t=1.0$, there will be a distribution of electrons throughout the entire gap length, and, therefore, losses will always be present in the system. In particular, for times $1.0<t<1.5$, losses come from the electrons composing the main body of the distribution that reaches the anode; at time $t=1.5$, all electrons composing the main body are

finally lost, producing a local minimum in $N(t)$ at this time. For times $t > 1.5$, the electron-density distribution becomes continuous in space, as can be seen in the plot labeled (6) in Fig. 3, plots (5) and (6) in Fig. 4, and plots (2)–(4) in Fig. 5. In the long-time limit the electron population will tend to disappear in the subcritical case [Fig. (3)]; for the supercritical case, the population will always increase in time [Fig. (5)]; and, finally, for the critical case [Fig. (4)], the electron density will asymptotically tend to the exponential distribution in Eq. (37). These results are consistent, within the restriction of the model adopted in this work, with those obtained from other approaches [11].

V. ELECTRON DENSITY-DENSITY CORRELATION FUNCTION AND ELECTRON NUMBER VARIANCE

In this section we analyze the equation for the electron density-density correlation function $R(x',x'',t)$, Eq. (17), and obtain approximate solutions in some particular cases. Equation (17) is a partial integro-differential equation that we shall transform into an integral equation for $R(x',x'',t)$, which is easier to handle; to this end, we first define the correlation between the total-electron population and the electron density at a position x, $\langle \delta N(t)\delta n(x,t)\rangle$, as

$$F(x,t)\equiv\int_0^L dy\, R(x,y,t)\ , \tag{38}$$

which is one of the source terms in Eq. (17). Thus, a double Laplace transform in the x' and x'' variables in that equation gives

$$\frac{\partial}{\partial t}\tilde{\tilde{R}}(k',k'',t)-[2a-\mu(k'+k'')]\tilde{\tilde{R}}(k',k'',t) =a\tilde{\rho}(k'+k'',t)+bN(t)+b[\tilde{F}(k',t)+\tilde{F}(k'',t)]\ , \tag{39}$$

where the following has been defined:

$$\tilde{\tilde{R}}(k',k'',t)=\int_0^\infty dx'\int_0^\infty dx''e^{-(k'x'+k''x'')}R(x',x'',t)\ , \tag{40}$$

$$\tilde{F}(k,t)=\int_0^\infty dx\, F(x,t)e^{-kx}\ , \tag{41}$$

and

$$\tilde{\rho}(k,t)=\int_0^\infty dx\,\rho(x,t)e^{-kx}\ . \tag{42}$$

Assuming that $F(x,t)$ is a known function, the solution to Eq. (39) is simply given by

$$\tilde{\tilde{R}}(k',k'',t)=\tilde{\tilde{R}}(k',k'',0)e^{[2a-\mu(k'+k'')]t}+e^{2at}\int_0^t d\tau\, e^{-2a\tau}e^{-(k'+k'')\mu(t-\tau)} \times[a\tilde{\rho}(k'+k'',\tau)+bN(\tau)+b\tilde{F}(k',\tau)+b\tilde{F}(k'',\tau)]\ . \tag{43}$$

Finally, taking the inverse Laplace transform in k' and k'', we get an expression for the density-density correlation function,

$$\begin{aligned}R(x',x'',t)=&e^{2at}R(x'-\mu t,x''-\mu t,0)\theta(x'-\mu t)\theta(x''-\mu t)\\&+\delta(x'-x'')\rho(x'',t)\{(e^{at}-1)\theta(x''-\mu t)+(e^{ax''/\mu}-1)\theta(t-x''/\mu)\}\\&+b\int_0^t d\tau\, e^{2a\tau}N(t-\tau)\delta(x'-\mu\tau)\delta(x''-\mu\tau)+\frac{b}{\mu}e^{2ax''/\mu}F(x'-x'',t-x''/\mu)\theta(x'-x'')\theta(t-x''/\mu)\\&+\frac{b}{\mu}e^{2ax'/\mu}F(x''-x',t-x'/\mu)\theta(x''-x')\theta(t-x'/\mu)\ .\end{aligned} \tag{44}$$

This is not a closed equation for the density-density correlation R, since it still depends on $F(x,t)$, which is a function of R itself. However, it can be integrated over the gap length to obtain a simpler integral equation for $F(x,t)$, namely,

$$\begin{aligned}F(x,t)=&e^{2at}\theta(L/\mu-t)\theta(x-\mu t)\int_0^{L-\mu t}dy\, R(x,y,0)+\rho(x,t)\{(e^{at}-1)\theta(x-\mu t)+(e^{ax/\mu}-1)\theta(t-x/\mu)\}\\&+\frac{b}{\mu}e^{2ax/\mu}N(t-x/\mu)\theta(t-x/\mu)+\frac{b}{\mu}e^{2ax/\mu}\theta(x/\mu-t)\int_{x-\mu t}^x dy\, e^{-2ay/\mu}F\left[y,t-\frac{x-y}{\mu}\right]\\&+\frac{b}{\mu}e^{2ax/\mu}\theta(t-x/\mu)\left\{\int_0^x dy\, e^{-2ay/\mu}F\left[y,t-\frac{x-y}{\mu}\right]+\int_0^{L-x}dy\, F(y,t-x/\mu)\right\}\ .\end{aligned} \tag{45}$$

An exact analytical solution to Eq. (45) is not possible and we have to restrict ourselves to some particular cases. In what follows we assume that the initial-electron distribution is deterministic, i.e., $R(x,y,0)=0$, so that $F(x,0)=0$.

Long-time limit

In the long-time limit, Eq. (45) reduces to

$$F(x)=(e^{ax/\mu}-1)\rho_{\mathrm{LT}}(x)+\frac{b}{\mu}e^{2ax/\mu}N_{\mathrm{LT}} + \frac{b}{\mu}e^{2ax/\mu}\left[\int_0^x dy\, e^{-2ay/\mu}F(y)+\int_0^{L-x}dy\, F(y)\right], \tag{46}$$

where $\rho_{\mathrm{LT}}(x)$, and N_{LT} denote the asymptotic density distribution and total-electron population, respectively. The solution of this equation may or may not exist depending on the values of the parameters a and b, as discussed below.

In the case when the parameters of the system satisfy the breakdown criterion, Eq. (35), the above integral equation becomes

$$F(x)=\frac{b}{\mu}N_{ss}e^{ax/\mu}(2e^{ax/\mu}-1) + \frac{b}{\mu}e^{2ax/\mu}\left[\int_0^x dy\, e^{-2ay/\mu}F(y)+\int_0^{L-x}dy\, F(y)\right]. \tag{47}$$

This equation can be shown to have no solution, i.e., $F(x,t)\to\infty$ as $t\to\infty$ (see Appendix B). It follows, then, that although the behavior of the ensemble averages of the electron number density and the total number of electrons tends to a finite stationary value, the fluctuations around their mean behavior grow in time, and the probability for a given realization to have no electron population will tend to 1 as time grows. In other words, given an ensemble of an infinite number of macroscopically identical systems, the fraction of realizations with almost no population at all will be increasing in time, while the fraction of realizations with high population values will be decreasing in such a way that the ensemble average tends to a finite constant. Thus, the breakdown criterion is meaningful only in relation to the mean electron population, since it does not imply stationarity of the fluctuations. This conclusion agrees with that of previous authors, for the same conditions [2,5,7].

When the parameters of the system satisfy the inequality $b(e^{aL/\mu}-1)/a<1$, it follows from Sec. IV that the mean electron population will disappear, i.e., $\rho_{\mathrm{LT}}(x)$ and $N_{\mathrm{LT}}\to 0$. The integral equation (46) reduces then to

$$F(x)=\frac{b}{\mu}e^{2ax/\mu}\left[\int_0^x dy\, e^{-2ay/\mu}F(y)+\int_0^{L-x}dy\, F(y)\right], \tag{48}$$

whose solution is the trivial solution $F(x)=0$, $\forall x$ (see Appendix B). Thus, the fluctuations of the electron population in the discharge will decay in time, following the mean behavior.

Finally, when $b(e^{aL/\mu}-1)/a>1$, both the mean electron density and total population will grow in time, and the long-time solution of Eq. (45) will not exist; however, since the fluctuations around these mean values are always increasing in time, it follows that the probability that the discharge will be extinguished at a given time will be always finite.

Time-dependent solutions

Time-dependent solutions of Eq. (45) are available by using the method of successive approximations [19] only in the subcritical case, where convergence of this method can be proved (see Appendix C). By introducing the dimensionless smallness parameter

$$\epsilon\equiv bL/\mu, \tag{49}$$

Eq. (45) can be cast as

$$F(x,t)=\rho(x,t)\{(e^{at}-1)\theta(x-\mu t)+(e^{ax/\mu}-1)\theta(t-x/\mu)\} + \epsilon e^{2ax/\mu}L^{-1}\left\{N(t-x/\mu)\theta(t-x/\mu)+\theta(x/\mu-t)\int_{x-\mu t}^{x}dy\, e^{-2ay/\mu}F\left[y,t-\frac{x-y}{\mu}\right] + \theta\left[t-\frac{x}{\mu}\right]\left[\int_0^x dy\, e^{-2ay/\mu}F\left[y,t-\frac{x-y}{\mu}\right]+\int_0^{L-x}dy\, F(y,t-x/\mu)\right]\right\}. \tag{50}$$

The solution, then, can be written as a power series of the smallness parameter, as

$$F(x,t)=\sum_{n=0}^{\infty}\epsilon^n F_n(x,t). \tag{51}$$

We shall point out here that the rigorous form of rearranging Eq. (50) in powers of the smallness parameter ϵ consists in expression the density $\rho(x,t)$ and the total number of electrons $N(t)$ also in powers of ϵ; however, as follows from Eqs. (31) and (32), besides being cumbersome, this procedure does not add anything but formality to our proposed solution (51). Thus, we shall assume that $\rho(x,t)$ and $N(t)$ are known functions of time and independent of ϵ.

Substitution of Eq. (51) in (50) yields the following recursive set of equations for the coefficients of the different powers of ϵ:

$$F_0(x,t)=\rho(x,t)\{(e^{at}-1)\theta(x-\mu t)+(e^{ax/\mu}-1)\theta(t-x/\mu)\}\ , \tag{52}$$

$$F_1(x,t)=e^{2ax/\mu}L^{-1}\left\{N(t-x/\mu)\theta(t-x/\mu)+\theta\left[\frac{x}{\mu}-t\right]\int_{x-\mu t}^{x}dy\,e^{-2ay/\mu}F_0\left[y,t-\frac{x-y}{\mu}\right]\right.$$
$$\left.+\theta\left[t-\frac{x}{\mu}\right]\left[\int_0^x dy\,e^{-2ay/\mu}F_0\left[y,t-\frac{x-y}{\mu}\right]+\int_0^{L-x}dy\,F_0(y,t-x/\mu)\right]\right\}\ , \tag{53}$$

and, for $m\geq 1$,

$$F_{m+1}(x,t)=e^{2ax/\mu}L^{-1}\left\{\theta\left[\frac{x}{\mu}-t\right]\int_{x-\mu t}^{x}dy\,e^{-2ay/\mu}F_m\left[y,t-\frac{x-y}{\mu}\right]\right.$$
$$\left.+\theta\left[t-\frac{x}{\mu}\right]\left[\int_0^x dy\,e^{-2ay/\mu}F_m\left[y,t-\frac{x-y}{\mu}\right]+\int_0^{L-x}dy\,F_m(y,t-x/\mu)\right]\right\}\ . \tag{54}$$

It is shown in Appendix C that the above expansion converges when

$$|\epsilon|<\frac{1}{2}\left[\frac{\mu}{4aL}\left[\frac{\mu}{4aL}[e^{4aL/\mu}-1]-1\right]\right]^{-1/2}\ . \tag{55}$$

Since we cannot find $F(x,t)$ exactly, we shall be content to find an approximate solution up to a reasonable order in ϵ.

Using the recursive equations (52)–(54) for $F(x,t)$, several quantities of interest can be approximately computed. Such is the case of the density-density correlation at the cathode and the anode position, which has a simple relation with $F(L,t)$, i.e.,

$$R(0,L;t)=\frac{b}{\mu}F(L,t)\ , \tag{56}$$

as follows from Eq. (44). Also, the variance of the total-electron population, $\sigma^2(t)\equiv\langle\delta N^2(t)\rangle$, can be obtained from $F(x,t)$, as the integral

$$\sigma^2(t)=\int_0^L dx\,F(x,t)\ ; \tag{57}$$

in particular, substituting Eq. (51) in the above expression, the variance is now expressed in powers of the smallness parameter,

$$\sigma^2(t)=\sum_{m=0}^{\infty}\epsilon^m\sigma_m^2(t)\ , \tag{58}$$

where

$$\sigma_m^2(t)\equiv\int_0^L dx\,F_m(x,t)\ . \tag{59}$$

The above results are illustrated in Figs. 6–8 for a subcritical discharge satisfying the convergence condition, Eq. (55). Once more, all quantities are expressed in the dimensionless form used earlier, and again the initial-electron distribution is assumed to be $\rho(x,0)=N_0\delta(x-x_0)$, with $x_0=0.5$.

Figures 6 and 7 show the time behavior of the mean electron population, as well as the standard deviation of

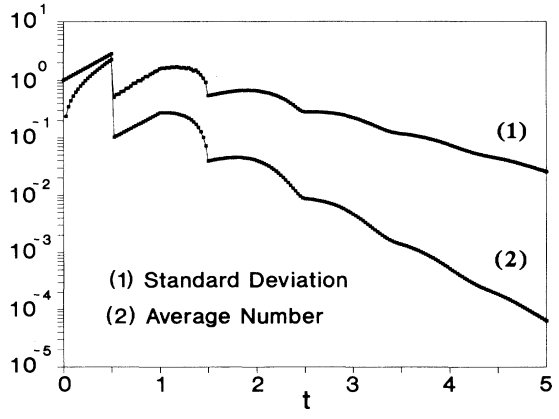

FIG. 6. Time behavior of the standard deviation $\sigma(t)$ and the average electron population for the subcritical discharge ($a=2.0$, $b=0.07$) of Fig. 2. The standard deviation is obtained from Eqs. (58) and (59), keeping terms up to order ϵ^2. Plot (1) is normalized by a factor $N_0^{-1/2}$ and plot (2) by a factor N_0^{-1}. Dimensionless quantities are used, as defined in the text.

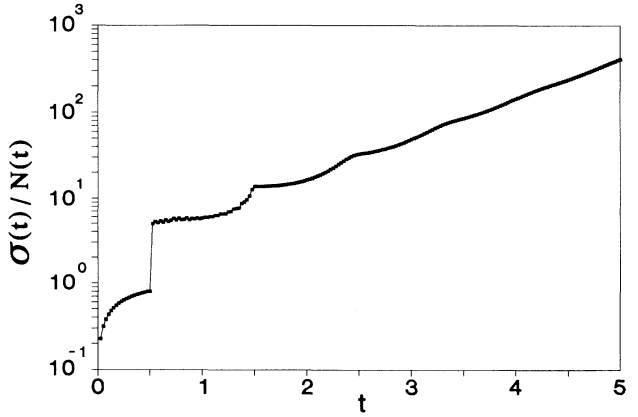

FIG. 7. Relative width of the fluctuations for the subcritical conditions in Fig. 6, normalized by a factor $N_0^{-1/2}$. Dimensionless quantities are used, as defined in the text.

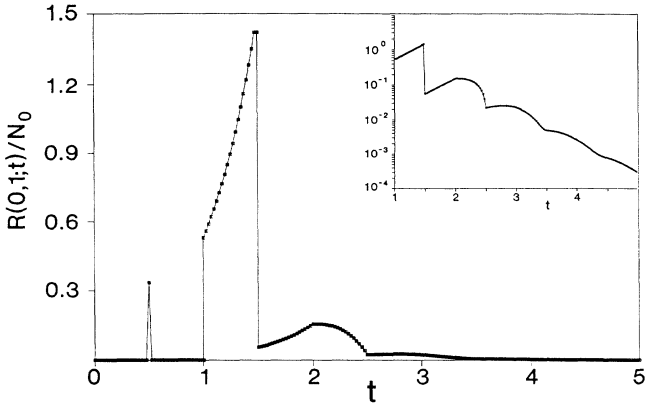

FIG. 8. Time behavior of the density-density correlation function at $x=0.0$ and $x=1.0$ for the subcritical conditions in Fig. 6, as obtained from Eqs. (51) and (56) keeping terms up to order ϵ^3. Dimensionless quantities are used, as defined in the text.

the fluctuations computed numerically from Eq. (58) up to order ϵ^2. Although both quantities tend to vanish in time, it can be observed in Fig. 7 that their ratio grows without limit. In this case, since $N(t)$ is a non-negative quantity, as time increases, fluctuations around the vanishing average $\langle N(t)\rangle$ give rise to more and more realizations with zero population. Thus, a given discharge is expected to become extinguished within few transit times L/μ, regardless of how large the initial population N_0 is.

Figure 8 shows the correlation of the density fluctuations near the cathode and the anode, $R(0,1;t)$, as a function of time, calculated from Eqs. (51) and (56) up to order ϵ^3. Since secondary production processes are always present in the system, the density-density correlation $R(0,1;t)$ is expected to be zero only when there are no electrons near the anode. As shown in the figure, this happens for times $0<t<1.0$, except at $t=0.5$ when the initial δ pulse reaches the anode [see Fig. 4]. It can be observed that the relative maxima of $R(0,1;t)$ appear whenever there is a relative maximum in the average electron density near the anode; this may be expected due to the nature of the secondary ionization processes and the fact that the density fluctuations are larger for the larger values of the average electron density, as described by Eq. (44).

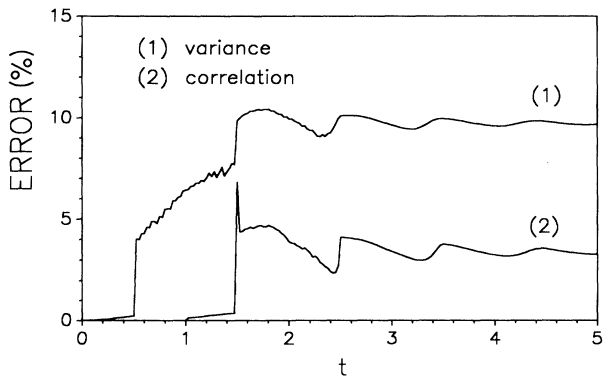

FIG. 9. Estimation of the upper bounds for the relative errors introduced in the calculation of the variance $\sigma^2(t)$ and the density-density correlation function $R(0,1;t)$ of Figs. 6–8 as a function of time. Dimensionless quantities are used, as defined in the text.

Finally, in Fig. 9 we show an estimation of the relative errors introduced by neglecting the term of order ϵ^4 of the expansion (51) in Eq. (56) for $R(0,1;t)$, and the term of order ϵ^3 in the expansion (58) for $\sigma^2(t)$. The plots indicate that the numerical results in Figs. 6–8 are accurate within a 7% error for the correlation and an 11% error for the variance.

VI. CONCLUDING REMARKS

This work illustrates the use of the characteristic functional to study the space-time stochastic fluctuations in the ionization growth in an electric discharge. Similar to the method of compounding moments, this technique discretizes the configuration space in cells to obtain a master equation, from where an equation for the characteristic function is obtained; taking the limit to the continuum, we derive an equation for the characteristic functional. In addition to avoiding the use of a probability in function space, this method has the advantage that the motion in the configuration space, as well as the boundary conditions when the system is finite, appear naturally in the equations for the moments; furthermore, in this approach the limit to the continuum is only performed when deriving the equation for the characteristic functional, while in the method of compounding moments, the limit to the continuum has to be taken in each one of the derived equations for the moments. The equations describing the evolution of quantities such as the average density and the density-density correlation function are obtained from the equation for the characteristic functional simply by taking appropriate functional derivatives.

The simple model considered here leads to an equation for the electron mean density whose analytical solution can be obtained exactly. From the analysis of this equation, the Townsend breakdown criterion is derived, and its physical meaning explained. Numerical examples are used here to illustrate that the electron mean density may increase, decrease, or tend to a finite stationary distribution, when the voltage between the electrodes is, respectively, above, below, or equal to the critical voltage at which the breakdown criterion is satisfied.

From the equation for the density-density correlation function, the fluctuations around the mean behavior of the electron population can be studied. It is found that the fluctuations around the mean behavior in a discharge satisfying the breakdown criterion always increase in time, from which it is concluded that the probability to find a particular discharge extinguished at a given time tends to 1, as time evolves. When the voltage between the electrodes exceeds the threshold voltage at which the breakdown criterion is satisfied, the fluctuations around the growing mean also increase in time. Finally, for discharges below the Townsend criterion, both the mean

and the fluctuations tend to vanish asymptotically. In some special cases of this latter situation, the method of successive approximations is guaranteed to converge, and it is used to obtain, by means of numerical integration, approximate solutions for the density-density correlation and the variance of the total number of electrons, as functions of time. It is concluded that the breakdown criterion is meaningful only in relation to the behavior of the mean electron population since, due to the multiplicative nature of the ionization process, fluctuations do not reach a stationary regime; ergodic properties cannot be used, then, to relate ensemble averages of the stochastic fluctations to temporal averages from a single realization.

In the absence of an external source of electrons, e.g., illumination of the cathode by ultraviolet light, stationary discharges are observed experimentally due, in part, to the stabilizing effects of the external circuit coupled to the gap; in fact, if the discharge gap is connected to a constant supply voltage through a resistor, when the current in the gap increases, the gap voltage decreases, and vice versa [5]. In addition, the role of nonlinearities, such as the effect of the space charge on the drift velocity and the ionization parameters, cannot be neglected when the electron population has become significant [11].

Extensions to this work, which include diffusion in addition to the constant drift, the ion population, and an external source of electrons produced by illuminating the cathode with ultraviolet light, are currently in progress. It is expected that the relation between theoretical results and experimental observations can be done more easily for subcritical discharges in the presence of ultraviolet illumination of the cathode, since in this case both the mean populations and the fluctuations will tend asymptotically to stationary finite distributions.

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APPENDIX A

Here, we first determine the region of the complex plane where expansion (30) is valid, that is, the region where the inequality

$$\left| \frac{b}{s-a-b} e^{-(s-a)L/\mu} \right| < 1 \tag{A1}$$

is satisfied. To shorten notation, we define the real parameters

$$\tau \equiv L/\mu \ , \quad d \equiv be^{a\tau} \ , \quad c \equiv a+b \ ; \tag{A2}$$

writing $s = s_r + is_i$, the above inequality becomes

$$de^{-s_r\tau}/\sqrt{(s_r-c)^2+s_i^2} < 1 \ . \tag{A3}$$

The maximum value of the LHS in this inequality is reached when $s_i = 0$; therefore, it is sufficient to ask

$$de^{-s_r\tau}/|s_r-c| < 1 \ . \tag{A4}$$

In the region $s_r > c$, this inequality becomes

$$de^{-s_r\tau} < s_r - c \ , \tag{A5}$$

which is satisfied when $s_r > s_0$, where s_0 is the solution to

$$de^{-s_0\tau} = s_0 - c \ . \tag{A6}$$

The analysis of the validity of the expansion for $s_r < c$ is irrelevant to our purpose. It is concluded that one of the regions of the complex plane where the expansion (30) is valid is the region where the real part of s is greater than s_0.

In what follows, it is shown that all the singularities of $\tilde{\rho}(x,s)$ in Eq. (29) lie to the left of s_0. In order for the inverse Laplace transform [18]

$$\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} ds \, e^{st} \tilde{\rho}(x,s) \tag{A7}$$

to be valid, the existence of a finite value of γ such that all the singularities of $\tilde{\rho}(x,s)$ lie to the left of it is necessary. From Eq. (30), the singularities of $\tilde{\rho}(x,s)$ satisfy

$$s - c + de^{-s\tau} = 0 \ , \tag{A8}$$

where we have used the definitions (A2). Substituting $s = s_r + is_i$ and equating to zero the real and imaginary parts, we get the following two coupled equations:

$$s_r - c + de^{-s_r\tau} \cos s_i\tau = 0 \tag{A9}$$

and

$$s_i - de^{-s_r\tau} \sin s_i\tau = 0 \ ; \tag{A10}$$

from these equations we obtain

$$(s_r-c)^2 + s_i^2 = d^2 e^{-2s_r\tau} \ . \tag{A11}$$

Solving for s_i the above equation and imposing the requirement that it must be real, we get

$$d^2 e^{-2s_r\tau} > (s_r-c)^2 \ . \tag{A12}$$

In the region $s_r > c$, Eq. (A12) becomes

$$de^{-s_r\tau} > s_r - c \ , \tag{A13}$$

from which it follows that the singularity exists in the interval $c < s_r < s_0$, where s_0 is the solution to Eq. (A6). In the region $s_r < c$, the singularities lie in those intervals where the inequality $de^{-s_r\tau} > c - s_r$ is satisfied; a detailed analysis of these last two intervals is irrelevant to our purpose. We conclude that the expansion in Eq. (30) is valid and the inverse Laplace transform can be obtained by choosing $\gamma > s_0$ in Eq. (A7).

APPENDIX B

In this appendix we use the Fredholm's alternative to show that there is no solution to Eq. (47) and that the solution to Eq. (48) is the trivial solution.

To this end, we first notice that these equations can be

written in the standard form

$$F(x)=g(x)+\frac{b}{\mu}e^{2ax/\mu}\left[\int_0^x dy\, e^{-2ay/\mu}F(y)+\int_0^{L-x}dy\, F(y)\right], \tag{B1}$$

where $g(x)$, known as the inhomogeneous term, is equal to $(bN_{ss}/\mu)e^{ax/\mu}[2e^{ax/\mu}-1]$ in the case of Eq. (47) and equal to zero in the case of Eq. (48). The Fredholm's alternative [20] is as follows: "Either the inhomogeneous integral equation is soluble, regardless of the inhomogeneous term, or else the corresponding homogeneous equation has a nontrivial solution."

In order to use this criterion, we convert Eq. (B1) into the following ordinary differential equation:

$$\frac{d^2}{dx^2}F(x)-\frac{2a}{\mu}\frac{d}{dx}F(x)+\frac{b^2}{\mu^2}\left[e^{2aL/\mu}-1-\frac{2a}{b}\right]F(x)=0\ , \tag{B2}$$

where the inhomogeneous term $g(x)$ has not been included. To study the existence of the solution to Eq. (47), we substitute the stationarity condition, Eq. (35), in the above equation to get

$$\frac{d^2F}{dx^2}-\frac{2a}{\mu}\frac{dF}{dx}+\frac{a^2}{\mu^2}F=0\ , \tag{B3}$$

whose general solution is

$$F(x)=C_1e^{ax/\mu}+C_2xe^{ax/\mu}\ . \tag{B4}$$

Substituting this expression in Eq. (B1) with $g(x)=0$, we find that $C_2\equiv 0$ and C_1 is undetermined, so that the general solution to the homogeneous equation is not trivial and it follows from the Fredholm's alternative that Eq. (47) has no solution.

On the other hand, for the integral equation (48) the general solution to (B2) is

$$F(x)=e^{ax/\mu}(C_1e^{\kappa x}+C_2e^{-\kappa x})\ , \tag{B5}$$

with

$$\kappa^2=1-\frac{b^2}{a^2}\left[e^{2aL/\mu}-1-\frac{2a}{b}\right], \tag{B6}$$

where it is easily proved that $\kappa^2>0$ for a subcritical system. After substituting the above expressions in Eq. (B1), with $g(x)=0$, it is easily verified that the only possible solution requires $C_1\equiv 0$, $C_2\equiv 0$.

APPENDIX C

Here we prove that the solution to Eq. (50) obtained by means of successive approximations, Eq. (51), converges whenever the parameter ϵ is small enough. To this end, we notice that Eq. (50) can be written as a system of coupled integral equations of the form

$$f_1(x,t)=g_1(x,t)+\epsilon\int_0^L dy\, K_{11}(x,y,t)f_1(y,t-x/\mu)+\epsilon\int_0^L dy\, K_{12}(x,y,t)f_2(y,t-x/\mu) \tag{C1}$$

and

$$f_2(x,t)=g_2(x,t)+\epsilon\int_0^L dy\, K_{21}(x,y,t)f_1(y,t)+\epsilon\int_0^L dy\, K_{22}(x,y,t)f_2(y,t)\ , \tag{C2}$$

where we have defined

$$f_1(x,t)=F(x,t)\ , \tag{C3}$$

$$f_2(x,t)=F(x,t+x/\mu)\ , \tag{C4}$$

$$g_1(x,t)=\rho(x,t)\{(e^{at}-1)\theta(x-\mu t)+(e^{ax/\mu}-1)\theta(t-x/\mu)\}+\epsilon e^{2ax/\mu}L^{-1}N(t-x/\mu)\theta(t-x/\mu)\ , \tag{C5}$$

$$g_2(x,t)=\rho(x,t+x/\mu)(e^{ax/\mu}-1)+\epsilon e^{2ax/\mu}L^{-1}N(t)\ , \tag{C6}$$

$$K_{11}(x,y,t)=L^{-1}e^{2ax/\mu}\theta(L-x-y)\theta(\mu t-x)\ , \tag{C7}$$

$$K_{12}(x,y,t)=L^{-1}e^{2a(x-y)/\mu}\theta(x-y)\times\{\theta(x-\mu t)\theta(y-x+\mu t)+\theta(\mu t-x)\}\ , \tag{C8}$$

$$K_{21}(x,y,t)=L^{-1}e^{2ax/\mu}\theta(L-x-y)\ , \tag{C9}$$

and

$$K_{22}(x,y,t)=L^{-1}e^{2a(x-y)/\mu}\theta(x-y)\ . \tag{C10}$$

Since all functions defined in Eqs. (C5)–(C10) are nonnegative, then, for a fixed time t, we consider the following system of coupled integral equations:

$$f_1^*(x)=g_1^{\max}(x)+\epsilon\int_0^L dy\, K_{11}^{\max}(x,y)f_1^*(y)+\epsilon\int_0^L dy\, K_{12}^{\max}(x,y)f_2^*(y) \tag{C11}$$

and

$$f_2^*(x)=g_2^{\max}(x)+\epsilon\int_0^L dy\, K_{21}^{\max}(x,y)f_1^*(y)+\epsilon\int_0^L dy\, K_{22}^{\max}(x,y)f_2^*(y)\ , \tag{C12}$$

where

$$g_1^{\max}(x),\ldots,K_{22}^{\max}(x,y)$$

are chosen to satisfy

$$g_1^{\max}(x)\geq g_1(x,t')\ \vdots\ K_{22}^{\max}(x,y)\geq K_{22}(x,y,t')\ , \tag{C13}$$

for $x,y\in[0,L]$ and $\forall t'\in[0,t]$. Proposing that

$$f_1(x,t)=\sum_{n=0}^{\infty}\epsilon^n f_1^{(n)}(x,t) \tag{C14}$$

$$\vdots$$

$$f_1^*(x)=\sum_{n=0}^{\infty}\epsilon^n f_1^{*(n)}(x) \qquad \text{(C15)}$$
$$\vdots ,$$

it is easily verified that $f_1^{(n)}(x,t)\le f_1^{*(n)}(x)$ and $f_2^{(n)}(x,t)\le f_2^{*(n)}(x)$. It follows then that if the expansions in Eq. (C15) converge, the expansions in Eq. (C14) also converge [21].

Successive approximations, Eq. (C15), converge uniformly if ϵ satisfies the inequality [22]

$$|\epsilon|<\left[\max_{i=1,2}\left\{\sum_{j=1}^{2}\left[\int_0^L dx\int_0^L dy\,|K_{ij}^{\max}(x,y)|^2\right]^{1/2}\right\}\right]^{-1}, \qquad \text{(C16)}$$

provided that the integrals

$$\int_0^L dy\,|K_{ij}^{\max}(x,y)|^2 \qquad \text{(C17)}$$

are bounded. Taking

$$K_{11}^{\max}(x,y)=K_{21}^{\max}(x,y)=L^{-1}e^{2ax/\mu}\theta(L-x-y) \qquad \text{(C18)}$$

and

$$K_{12}^{\max}(x,y)=K_{22}^{\max}(x,y)=L^{-1}e^{2a(x-y)/\mu}\theta(x-y)\ , \qquad \text{(C19)}$$

it follows that condition (C17) is satisfied; substituting (C18) and (C19) in (C16), we get that the convergence of successive approximations method is guaranteed whenever

$$|\epsilon|<\frac{1}{2}\left[\frac{\mu}{4aL}\left[\frac{\mu}{4aL}(e^{4aL/\mu}-1)-1\right]\right]^{-1/2}. \qquad \text{(C20)}$$

The above condition can be satisfied only for subcritical systems, i.e., $b(e^{aL/\mu}-1)/a<1$.

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